SIAM J. CONTROL OPTIM.
Vol. 48, No. 3, pp. 1542–1566

ERGODIC BSDES AND OPTIMAL ERGODIC CONTROL IN BANACH SPACES*

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Abstract. In this paper we introduce a new kind of backward stochastic differential equations, called ergodic BSDEs, which arise naturally in the study of optimal ergodic control. We study the existence, uniqueness, and regularity of solution to ergodic BSDEs. Then we apply these results to the optimal ergodic control of a Banach valued stochastic state equation. We also establish the link between the ergodic BSDEs and the associated Hamilton–Jacobi–Bellman equation. Applications are given to the optimal ergodic control of stochastic partial differential equations.

Key words. stochastic optimal control, backward stochastic differential equations, ergodic control, Hamilton–Jacobi–Bellman equation, controlled stochastic partial differential equations

AMS subject classifications. 60H10, 93E20

DOI. 10.1137/07069849X

1. Introduction.

In this paper we study the following type of (Markovian) backward stochastic differential equations with infinite horizon (which we shall call *ergodic* BSDEs or EBSDEs for short):

$$Y_t^x = Y_T^x + \int_t^T [\psi(X_\sigma^x, Z_\sigma^x) - \lambda]\, d\sigma - \int_t^T Z_\sigma^x dW_\sigma, \quad 0 \le t \le T < \infty. \tag{1.1}$$

In (1.1) $(W_t)_{t\ge 0}$ is a cylindrical Wiener process in a Hilbert space Ξ, X^x, is the solution (starting at x) of a forward stochastic differential equation with values in a Banach space E, and $\psi : E \times \Xi^* \to \mathbb{R}$ is a given function. Our aim is to find a triple (Y, Z, λ), where Y, Z are adapted processes taking values in $\mathbb{R}$ and Ξ^*, respectively, and λ is a real number, such that (1.1) is satisfied $\mathbb{P}$-a.s. for all $0 \le t \le T < \infty$. We stress the fact that λ is part of the unknowns of (1.1) and this is the reason why the above is a new class of BSDEs.

It is by now well known that BSDEs provide an efficient alternative tool to study optimal control problems; see, e.g., [20], [9] or, in an infinite dimensional framework, [12], [17]. But to the best of our knowledge, there exists no work in which BSDE techniques are applied to optimal control problems with *ergodic* cost functionals, that is, functionals depending only on the asymptotic behavior of the state (see, e.g., the cost defined in formula (1.4)). We mention here that the expression "ergodic BSDEs" has already been used in [4]. The notion of EBSDE introduced there is quite different from the one given in the present paper. Namely in [4], a different class of BSDEs is obtained by imposing a stationarity condition on the solution process. In particular,

*Received by the editors July 27, 2007; accepted for publication (in revised form) January 13, 2009; published electronically May 1, 2009. This work was supported by funds from the Marie Curie ITN grant, "Controlled Systems," GA 213841/2008. The first and third authors were supported by Italian Ministry PRIN 2006 "Equazioni di Kolmogorov."

http://www.siam.org/journals/sicon/48-3/69849.html

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no real number λ appears in BSDEs considered in [4], as the generators of their BSDEs are strictly monotone in y.

The purpose of the present paper is to show that BSDEs, in particular the class of EBSDEs mentioned above, are a very useful tool in the treatment of ergodic control problems as well, especially in an infinite dimensional framework.

There is a fairly large amount of literature dealing with optimal ergodic control problems for finite dimensional stochastic state equations by analytic techniques. We just mention the basic papers by Bensoussan and Frehse [2] and by Arisawa and Lions [1], where the problem is treated through the study of the corresponding Hamilton–Jacobi–Bellman (HJB) equation (solutions are understood in a classical sense and in a viscosity sense, respectively).

Concerning the infinite dimensional case it is known that both classical and viscosity notions of solutions are not so suitable concepts. Goldys and Maslowski in [15] employ a mild formulation of the HJB equation in a Hilbertian framework (see [5] and the references within for the corresponding mild formulations in the standard cases). In [15] the authors prove, by a fixed point argument that exploits the smoothing properties of the Ornstein–Uhlenbeck semigroup corresponding to the state equation, existence, and uniqueness of the solution of the stationary HJB equation for discounted infinite horizon costs. Then they pass to the limit, as the discount goes to zero, to obtain a mild solution of the HJB equation for the ergodic problem (see also [8]). Such techniques need to assume, besides a natural condition on the dissipativity of the state equation, nondegeneracy of the noise and a limitation on the Lipschitz constant (with respect to the gradient variable) of the Hamiltonian function. This last condition carries a bound on the size of the control domain (see [14] for similar conditions in the infinite horizon case).

The introduction of EBSDEs allows us to treat Banach valued state equations with general monotone nonlinear term and possibly degenerate noise. Nondegeneracy is replaced by a structure condition as it usually happens in BSDEs approach; see, for instance, [9], [12]. Roughly speaking one has to ask that the control is as degenerate as the noise in such a way that Girsanov transformation is applicable. We also notice that using L^∞ estimates specific to infinite horizon backward stochastic differential equations (see [3], [21], [16]) we are able to eliminate conditions on the Lipschitz constant of the Hamiltonian. On the other hand we will only consider bounded cost functionals.

To be more precise we consider a forward equation

$$dX_t^x = (AX_t^x + F(X_t^x))dt + GdW_t, \qquad X_0 = x,$$

where X has values in a Banach space E, F maps E to E, and A generates a strongly continuous semigroup of contractions. Appropriate dissipativity assumptions on $A+F$ ensure the exponential decay of the difference between the trajectories starting from different points $x, x' \in E$. We stress that this property plays a crucial role in our arguments.

Then we introduce the class of BSDEs with generator strictly monotone in y:

$$Y_t^{x,\alpha} = Y_T^{x,\alpha} + \int_t^T (\psi(X_\sigma^x, Z_\sigma^{x,\alpha}) - \alpha Y_\sigma^{x,\alpha})d\sigma - \int_t^T Z_\sigma^{x,\alpha} dW_\sigma, \quad 0 \le t \le T < \infty, \tag{1.2}$$

where $\alpha > 0$ and $\psi : E \times \Xi^* \to \mathbb{R}$ is bounded in the first variable and Lipschitz in the second (see [3], [21], or [16]). By estimates based on a Girsanov argument introduced in [3] we obtain uniform estimates on $\alpha Y^{x,\alpha}$ and $Y^{x,\alpha} - Y^{x',\alpha}$ that allow us to prove that, roughly speaking, $(Y^{x,\alpha} - Y_0^{0,\alpha}, Z^{x,\alpha}, \alpha Y_0^{0,\alpha})$ converge to a solution (Y^x, Z^x, λ) of the EBSDE (1.1) for all $x \in E$. We also show that λ is unique under very general conditions. On the contrary we cannot, in general, expect uniqueness of the solution to (1.1); see Remark 4.7. On the other hand, we show that we can find a solution of (1.1) with $Y_t^x = v(X_t^x)$ and $Z_t^x = \zeta(X_t^x)$, where v is Lipschitz and $v(0) = 0$. Moreover (v, ζ) turns out to be unique, at least in a special case where ψ is the Hamiltonian of a control problem and the processes X^x are recurrent (see section 8 where we adapt an argument from [15]). So it seems that the correct notion of uniqueness here is the uniqueness of the function v (and consequently of ζ); see the end of Remark 4.7.

If we further assume differentiability of F and ψ (in the Gateaux sense), then v is differentiable; moreover, $\zeta = \nabla v G$, and finally (v, λ) give a mild solution of the HJB equation

$$\mathcal{L}v(x) + \psi\left(x, \nabla v(x) G\right) = \lambda, \quad x \in E, \tag{1.3}$$

where linear operator $\mathcal{L}$ is formally defined by

$$\mathcal{L}f\left(x\right) = \frac{1}{2} Trace\left(GG^* \nabla^2 f\left(x\right)\right) + \langle Ax, \nabla f\left(x\right)\rangle_{E,E^*} + \langle F\left(x\right), \nabla f\left(x\right)\rangle_{E,E^*}.$$

Moreover if the Kolmogorov semigroup satisfies the smoothing property in Definition 5.6 and F is genuinely dissipative (see Definition 5.7), then v is bounded.

The above results are then applied to a control problem with cost

$$J(x, u) = \limsup_{T\to\infty} \frac{1}{T} \mathbb{E} \int_0^T L(X_s^x, u_s) ds, \tag{1.4}$$

where u is an adapted process (an admissible control) with values in a separable metric space U, and the state equation is a Banach valued evolution equation of the form

$$dX_t^x = (AX_t^x + F(X_t^x))\, dt + G(dW_t + R(u_t)\, dt),$$

where $R : U \to \Xi$ is bounded. It is clear that the above functional depends only on the asymptotic behavior of the trajectories of X^x. After appropriate formulation we prove that, setting $\psi(x, z) = \inf_{u\in U}[L(x, u) + zR(u)]$ in (1.1), λ is optimal; that is,

$$\lambda = \inf_u J(x, u),$$

where the infimum is over all admissible controls. Moreover Z allows us to construct an optimal feedback in the sense that

$$\lambda = J(x, u) \text{ if and only if } L(X_t^x, u_t) + Z_t R(u_t) = \psi(X_t^x, Z_t).$$

Finally (see section 9) we show that our assumptions allow us to treat ergodic optimal control problems for a stochastic heat equation with polynomial nonlinearity and space-time white noise. We notice that the Banach space setting is essential in order to treat nonlinear terms with superlinear growth in the state equation.

The paper is organized as follows. After a section on notation, we introduce the forward SDE; in section 4 we study the ergodic BSDEs; in section 5 we show, in addition, the differentiability of the solution assuming that the coefficient is Gateaux differentiable. In section 6 we study the ergodic HJB equation and apply our result to optimal ergodic control in section 7. Section 8 is devoted to showing the uniqueness of the solution, and the last section contains application of the ergodic control of a nonlinear stochastic heat equation.

2. Notation. Let E, F be Banach spaces and H a Hilbert space, all assumed to be defined over the real field and to be separable. The norms and the scalar product will be denoted $|\cdot|$, $\langle\cdot,\cdot\rangle$, with subscripts if needed. Duality between the dual space E^* and E is denoted $\langle\cdot,\cdot\rangle_{E^*,E}$. $L(E,F)$ is the space of linear bounded operators $E \to F$, with the operator norm. The domain of a linear (unbounded) operator A is denoted $D(A)$.

Given a bounded function $\phi : E \to \mathbb{R}$, we denote $\|\phi\|_0 = \sup_{x\in E}|\phi(x)|$. If, in addition, ϕ is also Lipschitz continuous, then $\|\phi\|_{\text{lip}} = \|\phi\|_0 + \sup_{x,x'\in E,\, x\neq x'}|\phi(x) - \phi(x')||x-x'|^{-1}$.

We say that a function $F : E \to F$ belongs to the class $\mathcal{G}^1(E,F)$ if it is continuous, has a Gateaux differential $\nabla F(x) \in L(E,F)$ at any point $x \in E$, and for every $k \in E$ the mapping $x \to \nabla F(x)k$ is continuous from E to F (i.e., $x \to \nabla F(x)$ is continuous from E to $L(E,F)$ if the latter space is endowed the strong operator topology). In connection with stochastic equations, the space $\mathcal{G}^1$ has been introduced in [12], to which we refer the reader for further properties.

Given a probability space $(\Omega, \mathcal{F}, \mathbb{P})$ with a filtration $(\mathcal{F}_t)_{t\geq 0}$, we consider the following classes of stochastic processes with values in a real separable Banach space K:

1. $L^p_{\mathcal{P}}(\Omega, C([0,T],K))$, $p \in [1,\infty)$, $T > 0$, is the space of predictable processes Y with continuous paths on $[0,T]$ such that
$$|Y|^p_{L^p_{\mathcal{P}}(\Omega,C([0,T],K))} = \mathbb{E}\sup_{t\in[0,T]}|Y_t|^p_K < \infty.$$
2. $L^p_{\mathcal{P}}(\Omega, L^2([0,T];K))$, $p \in [1,\infty)$, $T > 0$, is the space of predictable processes Y on $[0,T]$ such that
$$|Y|^p_{L^p_{\mathcal{P}}(\Omega,L^2([0,T];K))} = \mathbb{E}\left(\int_0^T |Y_t|^2_K\,dt\right)^{p/2} < \infty.$$
3. $L^2_{\mathcal{P},\text{loc}}(\Omega; L^2(0,\infty;K))$ is the space of predictable processes Y on $[0,\infty)$ that belong to the space $L^2_{\mathcal{P}}(\Omega, L^2([0,T];K))$ for every $T > 0$.

3. The forward equation. In a complete probability space $(\Omega, \mathcal{F}, \mathbb{P})$, we consider the following stochastic differential equation with values in a Banach space E:

$$\begin{cases} dX_t = AX_t dt + F(X_t)dt + GdW_t, & t \geq 0, \\ X_0 = x \in E. \end{cases} \tag{3.1}$$

We assume that E is continuously and densely embedded in a Hilbert space H, and that both spaces are real separable.

We will work under the following general assumptions.

HYPOTHESIS 3.1.

1. *The operator A is the generator of a strongly continuous semigroup of contractions in E. We assume that the semigroup $\{e^{tA},\, t \geq 0\}$ of bounded linear operators on E generated by A admits an extension to a strongly continuous semigroup of bounded linear operators on H that we denote by $\{S(t),\, t \geq 0\}$.*
2. *W is a cylindrical Wiener process in another real separable Hilbert space Ξ. Moreover by $\mathcal{F}_t$ we denote the σ-algebra generated by $\{W_s,\, s \in [0,t]\}$ and by the sets of $\mathcal{F}$ with $\mathbb{P}$-measure zero.*

3. *$F : E \to E$ is continuous and has polynomial growth (that is, there exist $c > 0, k \geq 0$ such that $|F(x)| \leq c(1 + |x|^k)$, $x \in E$). Moreover there exists $\eta > 0$ such that $A + F + \eta I$ is dissipative.*
4. *G is a bounded linear operator from Ξ to H. The bounded linear, positive, and symmetric operators on H defined by the formula*

$$Q_t h = \int_0^t S(s) G G^* S^*(s) h \, ds, \qquad t \geq 0, \; h \in H,$$

are assumed to be of trace class in H. Consequently we can define the stochastic convolution

$$W_t^A = \int_0^t S(t-s) G dW_s, \quad t \geq 0,$$

as a family of H-valued stochastic integrals. We assume that the process $\{W_t^A, \, t \geq 0\}$ admits an E-continuous version.

We recall that, for every $x \in E$, with $x \neq 0$, the subdifferential of the norm at x, $\partial(|x|)$, is the set of functionals $x^* \in E^*$ such that $\langle x^*, x \rangle_{E^*,E} = |x|$ and $|x^*|_{E^*} = 1$. If $x = 0$, then $\partial(|x|)$ is the set of functionals $x^* \in E^*$ such that $|x^*|_{E^*} \leq 1$. The dissipativity assumption on $A + F$ can be explicitly stated as follows: for $x, x' \in D(A) \subset E$ there exists $x^* \in \partial(|x - x'|)$ such that

$$\langle x^*, A(x - x') + F(x) - F(x') \rangle_{E^*,E} \leq -\eta \, |x - x'| \, .$$

We can state the following theorem (see, e.g., [6, Theorem 7.13] and [7, Theorem 5.5.13]).

THEOREM 3.2. *Assume that Hypothesis* 3.1 *holds true. Then for every $x \in E$* (3.1) *admits a unique mild solution, that is, an adapted E-valued process with continuous paths satisfying $\mathbb{P}$-a.s.*

$$X_t = e^{tA} x + \int_0^t e^{(t-s)A} F(X_s) \, ds + \int_0^t e^{(t-s)A} G dW_s, \qquad t \geq 0.$$

We denote the solution by X^x, $x \in E$.

Now we want to investigate the dependence of the solution on the initial datum.

PROPOSITION 3.3. *Under Hypothesis* 3.1 *it holds that*

$$|X_t^{x_1} - X_t^{x_2}| \leq e^{-\eta t} |x_1 - x_2| \, , \; t \geq 0, \;\; x_1, x_2 \in E.$$

Proof. Let $X_1(t) = X_t^{x_1}$ and $X_2(t) = X_t^{x_2}$, $x_1, x_2 \in E$. For $i = 1, 2$ we set $X_i^n(t) = J_n X_i(t)$, where $J_n = n(nI - A)^{-1}$. Since $X_i^n(t) \in D(A)$ for every $t \geq 0$, and

$$X_i^n(t) = e^{tA} J_n x_i + \int_0^t e^{(t-s)A} J_n F(X_i(s)) \, ds + \int_0^t e^{(t-s)A} J_n G dW_s,$$

we get

$$\frac{d}{dt} \left(X_1^n(t) - X_2^n(t) \right) = A \left(X_1^n(t) - X_2^n(t) \right) + J_n \left[F(X_1(t)) - F(X_2(t)) \right].$$

So, by Proposition II.8.5 in [23] $|X_1^n(t) - X_2^n(t)|$ also admits the left and right derivatives with respect to t, and there exists $x_n^*(t) \in \partial(|X_1^n(t) - X_2^n(t)|)$ such that the left derivative of $|X_1^n(t) - X_2^n(t)|$ satisfies the following equation

$$\frac{d^-}{dt}|X_1^n(t) - X_2^n(t)| = \left\langle x_n^*(t), \frac{d}{dt}(X_1^n(t) - X_2^n(t))\right\rangle_{E^*,E}.$$

So we have

$$\begin{aligned}\frac{d^-}{dt}|X_1^n(t) - X_2^n(t)| = {} & \langle x_n^*(t), A(X_1^n(t) - X_2^n(t)) \\ & +F(X_1^n(t)) - F(X_2^n(t))\rangle_{E^*,E} \\ & + \langle x_n^*(t), J_nF(X_1(t)) - F(X_1^n(t))\rangle_{E^*,E} \\ & - \langle x_n^*(t), J_nF(X_2(t)) - F(X_2^n(t))\rangle_{E^*,E} \\ \leq {} & -\eta|X_1^n(t) - X_2^n(t)| + |\delta_1^n(t) - \delta_2^n(t)|,\end{aligned}$$

where for $i = 1, 2$ we have set $\delta_i^n(t) = J_nF(X_i(t)) - F(X_i^n(t))$.

Multiplying the above by $e^{\eta t}$ we get

$$\frac{d^-}{dt}\left(e^{\eta t}|X_1^n(t) - X_2^n(t)|\right) \leq e^{\eta t}|\delta_1^n(t) - \delta_2^n(t)|.$$

We note that $\delta_i^n(t)$ tends to 0 uniformly in $t \in [0, T]$ for arbitrary $T > 0$. Indeed,

$$\delta_i^n(t) = nR(n, A)[F(X_i(t)) - F(X_i^n(t))] + (nR(n, A) - I)F(X_i(t)),$$

and the convergence to 0 follows by a classical argument (see, e.g., the proof of Theorem 7.10 in [6], since $X_i^n(t)$ tends to $X_i(t)$ uniformly in $t \in [0, T]$ and the maps $t \mapsto X_i(t)$ and $t \mapsto F(X_i(t))$ are continuous with respect to t).

Thus, letting $n \to \infty$, we can conclude

$$|X_1(t) - X_2(t)| \leq e^{-\eta t}|x_1 - x_2|,$$

and the claim is proved. ◻

We will also need the following assumptions.

Hypothesis 3.4. *We have* $\sup_{t\geq 0} \mathbb{E}|W_t^A|^2 < \infty$.

Hypothesis 3.5. $e^{tA}G(\Xi) \subset E$ *for all* $t > 0$ *and* $\int_0^{+\infty} |e^{tA}G|_{L(\Xi,E)}dt < \infty$.

We recall that for arbitrary Gaussian random variable Y with values in the Banach space E, the inequality

$$\mathbb{E}\,\phi(|Y| - \mathbb{E}\,|Y|) \leq \mathbb{E}\,\phi(2\sqrt{\mathbb{E}\,|Y|^2}\,\gamma)$$

holds for any convex nonnegative continuous function ϕ on E and for γ a real standard Gaussian random variable; see, e.g., [10, Example 3.1.2]. Upon taking $\phi(x) = |x|^p$, it follows that for every $p \geq 2$ there exists $c_p > 0$ such that $\mathbb{E}\,|Y|^p \leq c_p(\mathbb{E}\,|Y|^2)^{p/2}$. By the Gaussian character of W_t^A and the polynomial growth condition on F stated in Hypothesis 3.1, point 3, we see that Hypothesis 3.4 entails that for every $p \geq 2$

$$\sup_{t\geq 0}\mathbb{E}\left[|W_t^A|^p + |F(W_t^A)|^p\right] < \infty. \tag{3.2}$$

Proposition 3.6. *Under Hypothesis* 3.1 *it holds, for arbitrary* $T > 0$ *and arbitrary* $p \geq 1$, *that*

$$\mathbb{E}\sup_{t\in[0,T]}|X_t^x|^p \leq C_{p,T}(1 + |x|^p), \qquad x \in E. \tag{3.3}$$

If, in addition, Hypothesis 3.4 *holds, then, for a suitable constant* $C > 0$*,*

$$\sup_{t\geq 0} \mathbb{E}|X_t^x| \leq C(1+|x|), \qquad x \in E. \tag{3.4}$$

Moreover if, in addition, Hypothesis 3.5 *holds,* γ *is a bounded, adapted,* Ξ*-valued process and* $X^{x,\gamma}$ *is the mild solution of equation*

$$\begin{cases} dX_t^{x,\gamma} = AX_t^{x,\gamma}dt + F(X_t^{x,\gamma})dt + GdW_t + G\gamma_t\, dt, & t \geq 0, \\ X_0^{x,\gamma} = x \in E, \end{cases} \tag{3.5}$$

then it is still true that

$$\sup_{t\geq 0} \mathbb{E}|X_t^{x,\gamma}| \leq C_\gamma(1+|x|), \qquad x \in E, \tag{3.6}$$

for a suitable constant C_γ *depending only on a uniform bound for* γ*.*

Proof. We set $Z_t = X_t^x - W_t^A$, $Z_t^n = J_n Z_t$, then

$$\frac{d}{dt}Z_t^n = AZ_t^n + J_nF(X_t^x) = AZ_t^n + \left[F(Z_t^n + J_nW_t^A) - F(J_nW_t^A)\right] + F(W_t^A) + \delta_t^n,$$

where

$$\delta_t^n = J_nF(X_t^x) - F(J_nX_t^x) + F(J_nW_t^A) - F(W_t^A).$$

Proceeding as in the proof of Proposition 3.3 observing that, for all $t > 0$, $\int_0^t |\delta_s^n| ds \to 0$ as $n \to \infty$, we get

$$|Z_t| \leq e^{-\eta t}|x| + \int_0^t e^{-\eta(t-s)}|F(W_s^A)|ds, \quad \mathbb{P}-\text{a.s.}$$

and (3.4) follows from (3.2).

In the case in which X^x is replaced by $X^{x,\gamma}$ the proof is exactly the same just replacing W_t^A by $W_t^{A,\gamma} = W_t^A + \int_0^t e^{(t-s)A}G\gamma_s ds$.

Finally to prove (3.3) we notice that (see the discussion in [17]) the process W^A is a Gaussian random variable with values in $C([0,T],E)$. Therefore by the polynomial growth of F we get

$$\mathbb{E} \sup_{t\in[0,T]} \left[|W_t^A|^p + |F(W_t^A)|^p\right] \leq C_{p,T}(1+|x|^p),$$

and the claim follows as above. ∎

Finally the following result is proved exactly as Theorem 6.3.3. in [7].

THEOREM 3.7. *Assume that Hypotheses* 3.1 *and* 3.4 *hold, then* (3.1) *has a unique invariant measure in* E *that we will denote by* μ*. Moreover* μ *is strongly mixing (that is, for all* $x \in E$*, the law of* X_t^x *converges weakly to* μ *as* $t \to \infty$*). Finally there exists a constant* $C > 0$ *such that for any bounded Lipschitz function* $\phi : E \to \mathbb{R}$*,*

$$\left|\mathbb{E}\phi(X_t^x) - \int_E \phi\, d\mu\right| \leq C(1+|x|)e^{-\eta t/2}\|\phi\|_{lip}.$$

4. Ergodic BSDEs (EBSDEs). This section is devoted to the following type of BSDEs with infinite horizon:

$$Y_t^x = Y_T^x + \int_t^T [\psi(X_\sigma^x, Z_\sigma^x) - \lambda]\, d\sigma - \int_t^T Z_\sigma^x\, dW_\sigma, \quad 0 \le t \le T < \infty, \tag{4.1}$$

where λ is a real number and is part of the unknowns of the problem; the equation is required to hold for every t and T as indicated. On the function $\psi : E \times \Xi^* \to \mathbb{R}$ we assume the following hypothesis.

HYPOTHESIS 4.1. *There exist $K_x, K_z > 0$ such that*

$$|\psi(x, z) - \psi(x', z')| \le K_x|x - x'| + K_z|z - z'|, \qquad x, x' \in E,\ z, z' \in \Xi^*.$$

Moreover $\psi(\,\cdot\,, 0)$ is bounded. We denote $\sup_{x\in E} |\psi(x, 0)|$ by M.

We start by considering an infinite horizon equation with strictly monotonic drift, namely, for $\alpha > 0$, the equation

$$Y_t^{x,\alpha} = Y_T^{x,\alpha} + \int_t^T (\psi(X_\sigma^x, Z_\sigma^{x,\alpha}) - \alpha Y_\sigma^{x,\alpha}) d\sigma - \int_t^T Z_\sigma^{x,\alpha} dW_\sigma, \quad 0 \le t \le T < \infty. \tag{4.2}$$

The existence and uniqueness of solution to (4.2) under Hypothesis 4.1 was first studied by Briand and Hu in [3] and then generalized by Royer in [21]. They have established the following result when W is a finite-dimensional Wiener process but the extension to the case in which W is a Hilbert-valued Wiener process is immediate (see also [16]).

LEMMA 4.2. *Let us suppose that Hypotheses* 3.1 *and* 4.1 *hold. Then there exists a unique solution $(Y^{x,\alpha}, Z^{x,\alpha})$ to BSDE* (4.2) *such that $Y^{x,\alpha}$ is a bounded continuous process, and $Z^{x,\alpha}$ belongs to $L^2_{\mathcal{P},\mathrm{loc}}(\Omega; L^2(0, \infty; \Xi^*))$.*

Moreover $|Y_t^{x,\alpha}| \le M/\alpha$, $\mathbb{P}$-a.s. for all $t \ge 0$.

We define

$$v^\alpha(x) = Y_0^{\alpha,x}.$$

We notice that by the above $|v^\alpha(x)| \le M/\alpha$ for all $x \in E$. Moreover by the uniqueness of the solution of (4.2) it follows that $Y_t^{\alpha,x} = v^\alpha(X_t^x)$.

To establish Lipschitz continuity of v^α (uniformly in α) we use a Girsanov argument due to Briand and Hu; see [3]. Here and in the following we use an infinite-dimensional version of the Girsanov formula that can be found, for instance, in [6].

LEMMA 4.3. *Under Hypotheses* 3.1 *and* 4.1 *the following holds for any $\alpha > 0$:*

$$|v^\alpha(x) - v^\alpha(x')| \le \frac{K_x}{\eta}|x - x'|, \qquad x, x' \in E.$$

Proof. We briefly report the argument for the reader's convenience.

We set $\tilde{Y} = Y^{\alpha,x} - Y^{\alpha,x'}$, $\tilde{Z} = Z^{\alpha,x} - Z^{\alpha,x'}$,

$$\beta_t = \begin{cases} \dfrac{\psi(X_t^{x'}, Z_t^{\alpha,x'}) - \psi(X_t^{x'}, Z_t^{\alpha,x})}{|Z_t^{\alpha,x} - Z_t^{\alpha,x'}|^2_{\Xi^*}} \left(Z_t^{\alpha,x} - Z_t^{\alpha,x'}\right)^* & \text{if } Z_t^{\alpha,x} \ne Z_t^{\alpha,x'} \\ 0 & \text{elsewhere,} \end{cases}$$

$$f_t = \psi(X_t^x, Z_t^{x,\alpha}) - \psi(X_t^{x'}, Z_t^{x,\alpha}).$$

By Hypothesis 4.1, β is a bounded, Ξ-valued, adapted process; thus there exists a probability $\tilde{\mathbb{P}}$ under which $\tilde{W}_t = \int_0^t \beta_s ds + W_t$ is a cylindrical Ξ-valued Wiener process for $t \in [0,T]$. Then $(\tilde{Y}, \tilde{Z})$ verifies, for all $0 \le t \le T < \infty$,

$$\tilde{Y}_t = \tilde{Y}_T - \alpha \int_t^T \tilde{Y}_\sigma d\sigma + \int_t^T f_\sigma d\sigma - \int_t^T \tilde{Z}_\sigma d\tilde{W}_\sigma. \tag{4.3}$$

Computing $d(e^{-\alpha t}\tilde{Y}_t)$, integrating over $[0,T]$, estimating the absolute value, and finally taking the conditional expectation $\tilde{\mathbb{E}}^{\mathcal{F}_t}$ with respect to $\tilde{\mathbb{P}}$ and $\mathcal{F}_t$, we get

$$|\tilde{Y}_t| \le e^{-\alpha(T-t)}\tilde{\mathbb{E}}^{\mathcal{F}_t}|\tilde{Y}_T| + \tilde{\mathbb{E}}^{\mathcal{F}_t}\int_t^T e^{-\alpha(s-t)}|f_s|ds.$$

Now we recall that $\tilde{Y}$ is bounded and that $|f_t| \le K_x|X_t^x - X_t^{x'}| \le K_x e^{-\eta t}|x-x'|$ by Proposition 3.3. Thus if $T \to \infty$, we get $|\tilde{Y}_t| \le K_x(\eta+\alpha)^{-1}e^{\alpha t}|x-x'|$, and the claim follows setting $t=0$. ∎

By the above lemma if we set

$$\overline{v}^\alpha(x) = v^\alpha(x) - v^\alpha(0),$$

then $|\overline{v}^\alpha(x)| \le K_x\eta^{-1}|x|$ for all $x \in E$ and for all $\alpha > 0$. Moreover, by Lemma 4.2, $\alpha|v^\alpha(0)| \le M$.

Thus by a diagonal procedure we can construct a sequence $\alpha_n \searrow 0$ such that for all x in a countable dense subset $D \subset E$

$$\overline{v}^{\alpha_n}(x) \to \overline{v}(x), \qquad \alpha_n v^{\alpha_n}(0) \to \overline{\lambda}, \tag{4.4}$$

for a suitable function $\overline{v} : D \to \mathbb{R}$ and for a suitable real number $\overline{\lambda}$.

Moreover, by Lemma 4.3, $|\overline{v}^\alpha(x) - \overline{v}^\alpha(x')| \le K_x\eta^{-1}|x-x'|$ for all $x, x' \in E$ and all $\alpha > 0$. So $\overline{v}$ can be extended to a Lipschitz function defined on the whole E (with Lipschitz constant $K_x\eta^{-1}$) and

$$\overline{v}^{\alpha_n}(x) \to \overline{v}(x), \qquad x \in E. \tag{4.5}$$

THEOREM 4.4. *Assume Hypotheses* 3.1 *and* 4.1 *hold. Moreover let* $\bar{\lambda}$ *be the real number in* (4.4) *and define* $\bar{Y}_t^x = \bar{v}(X_t^x)$ *(where* $\overline{v}$ *is the Lipschitz function with* $\overline{v}(0) = 0$ *defined in* (4.5)*). Then there exists a process* $\overline{Z}^x \in L^2_{\mathcal{P},\mathrm{loc}}(\Omega; L^2(0,\infty;\Xi^*))$ *such that* $\mathbb{P}$*-a.s. the EBSDE* (4.1) *is satisfied by* $(\bar{Y}^x, \bar{Z}^x, \bar{\lambda})$ *for all* $0 \le t \le T$.

Moreover there exists a measurable function $\overline{\zeta} : E \to \Xi^*$ *such that* $\overline{Z}_t^x = \overline{\zeta}(X_t^x)$.

Proof. Let $\overline{Y}_t^{x,\alpha} = Y_t^{x,\alpha} - v^\alpha(0) = \overline{v}^\alpha(X_t^x)$. Clearly we have, $\mathbb{P}$-a.s.,

$$\overline{Y}_t^{x,\alpha} = \overline{Y}_T^{x,\alpha} + \int_t^T (\psi(X_\sigma^x, Z_\sigma^{x,\alpha}) - \alpha\overline{Y}_\sigma^{x,\alpha} - \alpha v^\alpha(0))d\sigma - \int_t^T Z_\sigma^{x,\alpha}dW_\sigma,\ 0 \le t \le T < \infty. \tag{4.6}$$

Since $|\bar{v}^\alpha(x)| \le K_x|x|/\eta$, inequality (3.3) ensures that $\mathbb{E}\sup_{t\in[0,T]}\left[\sup_{\alpha>0}|\overline{Y}_t^{x,\alpha}|^2\right] < +\infty$ for any $T > 0$. Thus, if we define $\overline{Y}^x = \overline{v}(X^x)$, then by dominated convergence theorem

$$\mathbb{E}\int_0^T |\overline{Y}_t^{x,\alpha_n} - \overline{Y}_t^x|^2 dt \to 0 \quad \text{and} \quad \mathbb{E}|\overline{Y}_T^{x,\alpha_n} - \overline{Y}_T^x|^2 \to 0$$

as $n \to \infty$ (where $\alpha_n \searrow 0$ is a sequence for which (4.4) and (4.5) hold).

We claim now that there exists $\overline{Z}^x \in L^2_{\mathcal{P},\mathrm{loc}}(\Omega; L^2(0,\infty;\Xi^*))$ such that

$$\mathbb{E}\int_0^T |Z_t^{x,\alpha_n} - \overline{Z}_t^x|^2_{\Xi^*}\,dt \to 0.$$

Let $\tilde{Y} = \bar{Y}^{x,\alpha_n} - \bar{Y}^{x,\alpha_m}$, $\tilde{Z} = Z^{x,\alpha_n} - Z^{x,\alpha_m}$. Applying Itô's rule to $\tilde{Y}^2$, we get by standard computations

$$\tilde{Y}_0^2 + \mathbb{E}\int_0^T |\tilde{Z}_t|^2_{\Xi^*}\,dt = \mathbb{E}\tilde{Y}_T^2 + 2\mathbb{E}\int_0^T \tilde{\psi}_t\tilde{Y}_t dt - 2\mathbb{E}\int_0^T [\alpha_n Y_t^{x,\alpha_n} - \alpha_m Y_t^{x,\alpha_m}]\,\tilde{Y}_t\,dt,$$

where $\tilde{\psi}_t = \psi(X_t^x, Z_t^{x,\alpha_n}) - \psi(X_t^x, Z_t^{x,\alpha_m})$. We notice that $|\tilde{\psi}_t| \le K_z|\tilde{Z}_t|$ and $\alpha_n|Y_t^{x,\alpha_n}| \le M$. Thus

$$\mathbb{E}\int_0^T |\tilde{Z}_t|^2_{\Xi^*}\,dt \le c\left[\mathbb{E}(\tilde{Y}_T)^2 + \mathbb{E}\int_0^T (\tilde{Y}_t)^2 dt + \mathbb{E}\int_0^T |\tilde{Y}_t| dt\right].$$

It follows that the sequence $\{Z^{x,\alpha_n}\}$ is Cauchy in $L^2(\Omega; L^2(0,T;\Xi^*))$ for all $T > 0$, and our claim is proved.

Now we can pass to the limit as $n \to \infty$ in (4.6) to obtain

$$\overline{Y}_t^x = \overline{Y}_T^x + \int_t^T (\psi(X_\sigma^x, \overline{Z}_\sigma^x) - \overline{\lambda})d\sigma - \int_t^T \overline{Z}_\sigma^x dW_\sigma, \quad 0 \le t \le T < \infty. \tag{4.7}$$

We notice that the above equation also ensures continuity of the trajectories of $\overline{Y}$. It remains now to prove that we can find a measurable function $\bar{\zeta} : E \to \Xi^*$ such that $\overline{Z}_t^x = \bar{\zeta}(X_t^x)$, $\mathbb{P}$-a.s. for almost every $t \ge 0$.

By a general argument (see, for instance, [11]) we know that for all $\alpha > 0$ there exists $\zeta^\alpha : E \to \Xi^*$ such that $Z_t^{x,\alpha} = \zeta^\alpha(X_t^x)$, $\mathbb{P}$-a.s. for almost every $t \ge 0$.

To construct $\bar{\zeta}$ we need some more regularity of the processes $Z^{x,\alpha}$ with respect to x.

If we compute $d(Y_t^{x,\alpha} - Y_t^{x',\alpha})^2$, we get by the Lipschitz character of ψ

$$\begin{aligned}\mathbb{E}\int_0^T |Z_t^{x,\alpha} - Z_t^{x',\alpha}|^2_{\Xi^*}\,dt &\le \mathbb{E}(v^\alpha(X_T^x) - v^\alpha(X_T^{x'}))^2 \\ &+ 2\mathbb{E}\int_0^T \left(K_x|X_s^x - X_s^{x'}| + K_z|Z_s^{x,\alpha} - Z_s^{x',\alpha}|\right)\left|v^\alpha(X_s^x) - v^\alpha(X_s^{x'})\right| ds.\end{aligned}$$

By the Lipschitz continuity of v^α (uniform in α), that of ψ and Proposition 3.3 we immediately get

$$\mathbb{E}\int_0^T |Z_t^{x,\alpha} - Z_t^{x',\alpha}|^2_{\Xi^*}\,dt \le c|x - x'|^2 \tag{4.8}$$

for a suitable constant c (that may depend on T).

Now we fix an arbitrary $T > 0$, and by a diagonal procedure (using separability of E) we construct a subsequence $(\alpha_n') \subset (\alpha_n)$ such that $\alpha_n' \searrow 0$ and

$$\mathbb{E}\int_0^T |Z_t^{x,\alpha_n'} - Z_t^{x,\alpha_m'}|^2_{\Xi^*}\,dt \le 2^{-n}$$

for all $m \geq n$ and for all $x \in E$. Consequently $Z_t^{x,\alpha'_n} \to \overline{Z}_t^x$, $\mathbb{P}$-a.s. for a.e. $t \in [0,T]$. Then we set

$$\bar{\zeta}(x) = \begin{cases} \lim_n \zeta^{\alpha'_n}(x) & \text{if the limit exists in } \Xi^*, \\ 0 & \text{elsewhere.} \end{cases}$$

Since $Z_t^{x,\alpha'_n} = \zeta^{\alpha'_n}(X_t^x) \to \overline{Z}_t^x$ $\mathbb{P}$-a.s. for a.e. $t \in [0,T]$ we immediately get that, for all $x \in E$, the process X_t^x belongs $\mathbb{P}$-a.s. for a.e. $t \in [0,T]$ to the set where $\lim_n \zeta^{\alpha'_n}(x)$ exists and consequently $\overline{Z}_t^x = \bar{\zeta}(X_t^x)$. □

Remark 4.5. We notice that the solution we have constructed above has the following "linear growth" property with respect to X: there exists $c > 0$ such that, $\mathbb{P}$-a.s.,

$$|\overline{Y}_t^x| \leq c|X_t^x| \text{ for all } t \geq 0. \tag{4.9}$$

If we require similar conditions, then we immediately obtain uniqueness of λ.

THEOREM 4.6. *Assume that, in addition to Hypotheses* 3.1, 3.4, *and* 4.1, *Hypothesis* 3.5 *holds as well. Moreover suppose that, for some $x \in E$, the triple (Y', Z', λ') verifies $\mathbb{P}$-a.s.* (4.1) *for all $0 \leq t \leq T$, where Y' is a progressively measurable continuous process, Z' is a process in $L^2_{\mathcal{P},\mathrm{loc}}(\Omega; L^2(0,\infty;\Xi^*))$, and $\lambda' \in \mathbb{R}$. Finally assume that there exists $c_x > 0$ (that may depend on x) such that $\mathbb{P}$-a.s.*

$$|Y'_t| \leq c_x(|X_t^x| + 1) \textit{ for all } t \geq 0.$$

Then $\lambda' = \bar{\lambda}$.

Proof. Let $\tilde{\lambda} = \lambda' - \lambda$, $\tilde{Y} = Y' - \overline{Y}^x$, $\tilde{Z} = Z' - \overline{Z}^x$. By easy computations

$$\tilde{\lambda} = T^{-1}\left[\tilde{Y}_T - \tilde{Y}_0\right] + T^{-1}\int_0^T \tilde{Z}_t \gamma_t dt - T^{-1}\int_0^T \tilde{Z}_t dW_t,$$

where

$$\gamma_t := \begin{cases} \dfrac{\psi(X_t^x, Z'_t) - \psi(X_t^x, \overline{Z}_t^x)}{|Z'_t - \overline{Z}_t^x|^2_{\Xi^*}} \left(Z'_t - \overline{Z}_t^x\right)^* & \text{if } Z'_t \neq \overline{Z}_t^x, \\ 0 & \text{elsewhere} \end{cases}$$

is a bounded Ξ-valued progressively measurable process. By the Girsanov theorem there exists a probability measure $\mathbb{P}_\gamma$ under which $W_t^\gamma = -\int_0^t \gamma_s ds + W_t$, $t \in [0,T]$, is a cylindrical Wiener process in Ξ. Thus computing expectation with respect to $\mathbb{P}_\gamma$ we get

$$\tilde{\lambda} = T^{-1}\mathbb{E}^{\mathbb{P}_\gamma}\left[\tilde{Y}_T - \tilde{Y}_0\right].$$

Consequently, taking into account (4.9), we get

$$|\tilde{\lambda}| \leq cT^{-1}\mathbb{E}^{\mathbb{P}_\gamma}(|X_T^x| + 1) + cT^{-1}(|x| + 1). \tag{4.10}$$

With respect to W^γ, X^x is the mild solution of

$$\begin{cases} dX_t^{x,\gamma} = AX_t^{x,\gamma}dt + F(X_t^{x,\gamma})dt + GdW_t^\gamma + G\gamma_t\,dt, & t \geq 0, \\ X_0^{x,\gamma} = x \in E, \end{cases}$$

and by (3.6) we get $\sup_{T\geq 0}\mathbb{E}^{\mathbb{P}_\gamma}|X_T^x| < \infty$. So if we let $T \to \infty$ in (4.10), we conclude that $\tilde{\lambda} = 0$. □

Remark 4.7. The solution to EBSDE (4.1) is, in general, not unique. It is evident that the equation is invariant with respect to the addition of a constant to Y, but we can also construct an arbitrary number of solutions that do not differ only by a constant (even if we require them to be bounded).

Indeed, consider the equation

$$-dY_t = [\psi(Z_t) - \lambda]dt - Z_t dW_t, \tag{4.11}$$

where W is a standard Brownian motion and $\psi : \mathbb{R} \to \mathbb{R}$ is bounded differentiable and has bounded derivative.

One solution is $Y = 0, Z = 0, \lambda = \psi(0)$ (without loss of generality we can suppose that $\psi(0) = 0$).

Let now $\phi : \mathbb{R} \to \mathbb{R}$ be an arbitrary bounded differentiable function with bounded derivative. The following BSDE on $[t, T]$ admits a solution:

$$\begin{cases} -dY_s^{x,t} &= \psi(Z_s^{x,t})ds - Z_s^{x,t} dW_s, \\ Y_T^{x,t} &= \phi(x + W_T - W_t). \end{cases}$$

If we define $u(t,x) = Y_t^{x,t}$, then both u and ∇u are bounded. Moreover if $\tilde{Y}_t = Y_t^{0,0} = u(t, W_t)$, $\tilde{Z}_t = Z_t^{0,0} = \nabla u(t, W_t)$, then

$$\begin{cases} -d\tilde{Y}_t &= \psi(\tilde{Z}_t)dt - \tilde{Z}_t dW_t, \quad t \in [0,T], \\ \tilde{Y}_T &= \phi(W_T). \end{cases}$$

Then it is enough to extend with $\tilde{Y}_t = \tilde{Y}_T$, $\tilde{Z}_t = 0$ for $t > T$ to construct a bounded solution to (4.11).

We notice that the above example is in a certain sense in non-Markovian framework. To be more specific, only one of the solutions constructed above (the trivial one) can be represented as $Y_t = v(X_t^x)$, $Z_t = \zeta(X_t^x)$ (where in this case $X_t^x = x + W_t$). Moreover in section 8 we show that if we impose that $Y_t = v(X_t^x)$, $Z_t = \zeta(X_t^x)$ for suitable measurable functions $v : E \to \mathbb{R}$ and $\zeta : E \to \Xi^*$, then, at least in a particular case, v, and consequently ζ, is unique. So the right question, still open in the general case, seems to be uniqueness of the function v rather than uniqueness of the process Y.

Remark 4.8. The existence result in Theorem 4.4 can be easily extended to the case of ψ satisfying only the conditions

$$|\psi(x,z) - \psi(x',z)| \le K_x|x - x'|, \quad |\psi(x,0)| \le M, \quad |\psi(x,z)| \le K_z(1 + |z|).$$

Indeed we can construct a sequence $\{\psi^n : n \in \mathbb{N}\}$ of functions Lipschitz in x and z such that for all $x, x' \in H$, $z \in \Xi^*$, $n \in \mathbb{N}$,

$$|\psi^n(x,z) - \psi^n(x',z)| \le K_x'|x - x'|, \quad |\psi^n(x,0)| \le M', \quad |\psi^n(x,z)| \le K_z'(1 + |z|),$$

$$\lim_{n\to\infty} |\psi^n(x,z) - \psi(x,z)| = 0.$$

This can be done by projecting x to the subspaces generated by a basis in Ξ^* and then regularizing by the standard mollification techniques; see [13]. We know that if $(\bar{Y}^{x,n}, \bar{Z}^{x,n}, \lambda_n)$ is the solution of the EBSDE (4.1) with ψ replaced by ψ^n, then $\bar{Y}_t^{x,n} = \bar{v}^n(X_t^x)$ with

$$|\bar{v}^n(x) - \bar{v}^n(x')| \le \frac{K_x'}{\eta}|x - x'|, \quad \bar{v}^n(0) = 0, \quad |\lambda_n| \le M'.$$

Thus we can assume (considering, if needed, a subsequence) that $\bar{v}^n(x) \to \bar{v}(x)$ and $\lambda_n \to \lambda$. The rest of the proof is similar to the one of Theorem 4.4.

5. Differentiability. We are now interested in the differentiability of the solution to the EBSDE (4.1) with respect to x.

THEOREM 5.1. *Assume that Hypotheses* 3.1 *and* 4.1 *hold. Moreover assume that* F *is of class* $\mathcal{G}^1(E,E)$ *with* ∇F *bounded on bounded sets of* E. *Finally assume that* ψ *is of class* $\mathcal{G}^1(E\times\Xi^*,E)$. *Then the function* $\overline{v}$ *defined in* (4.5) *is of class* $\mathcal{G}^1(E,\mathbb{R})$.

Proof. In [17] it is proved that for arbitrary $T>0$ the map $x\to X^x$ is of class $\mathcal{G}^1$ from E to $L^p_{\mathcal{P}}(\Omega,C([0,T],E))$. Moreover Proposition 3.3 ensures that for all $h\in E$,

$$|\nabla X_t^x h|\le e^{-\eta t}|h|,\quad \mathbb{P}\text{-a.s. for all } t\in[0,T]. \tag{5.1}$$

Under the previous conditions one can proceed exactly as in Theorem 3.1 of [16] to prove that for all $\alpha>0$ the map v^α is of class $\mathcal{G}^1$.

Then we consider again (4.2):

$$Y_t^{x,\alpha}=Y_T^{x,\alpha}+\int_t^T(\psi(X_\sigma^x,Z_\sigma^{x,\alpha})-\alpha Y_\sigma^{x,\alpha})d\sigma-\int_t^T Z_\sigma^{x,\alpha}dW_\sigma,\quad 0\le t\le T<\infty,$$

we recall that $Y_T^{x,\alpha}=v^\alpha(X_T^x)$, and apply again [17] (see Proposition 4.2 there) and [12] (see Proposition 5.2 there) to obtain that for all $\alpha>0$ the map $x\to Y^{x,\alpha}$ is of class $\mathcal{G}^1$ from E to $L^2_{\mathcal{P}}(\Omega,C([0,T],\mathbb{R}))$ and the map $x\to Z^{x,\alpha}$ is of class $\mathcal{G}^1$ from E to $L^2_{\mathcal{P}}(\Omega,L^2([0,T],\Xi^*))$. Moreover for all $h\in E$ it holds (for all $t\ge0$ since T was arbitrary) that

$$\begin{aligned}-d\nabla Y_t^{\alpha,x}h &= [\nabla_x\psi(X_t^x,Z_t^{\alpha,x})\nabla X_t^x h+\nabla_z\psi(X_t^x,Z_t^{\alpha,x})\nabla Z_t^{\alpha,x}h-\alpha\nabla Y_t^{\alpha,x}h]dt\\ &\quad -\nabla Z^{\alpha,x}h dW_t.\end{aligned}$$

We also know that $|Y_t^{\alpha,x}|\le M/\alpha$. Now we set

$$U_t^{\alpha,x}=e^{\eta t}\nabla Y_t^{\alpha,x}h,\quad V^{\alpha,x}=e^{\eta t}\nabla Z_t^{\alpha,x}h.$$

Then $(U^{\alpha,x},V^{\alpha,x})$ satisfies the following BSDE:

$$\begin{aligned}-dU_t^{\alpha,x} &= [e^{\eta t}\nabla_x\psi(X_t^x,Z_t^{\alpha,x})\nabla X_t^x h-(\alpha+\eta)U_t^{\alpha,x}+\nabla_z\psi(X_t^x,Z_t^{\alpha,x})V_t^{\alpha,x}]dt\\ &\quad -V_t^{\alpha,x}dW_t.\end{aligned}$$

By (5.1) and the usual Girsanov argument (recall the $\nabla_x\psi$ and $\nabla_z\psi$ are bounded),

$$|U_t^{\alpha,x}|\le\frac{c|h|}{\alpha+\eta}\ \forall t\ge0,\quad \mathbb{P}-\text{a.s.},\quad \text{i.e.,}\quad |\nabla Y_t^{x,\alpha}|\le e^{-\eta t}\frac{c|h|}{\alpha+\eta}.$$

Moreover, consider the limit equation, with unknown (U^x,V^x),

$$-dU_t^x=[e^{\eta t}\nabla_x\psi(X_t^x,\bar Z_t^x)\nabla X_t^x h-\eta U_t^x+\nabla_z\psi(X_t^x,\bar Z_t^x)V^x]dt-V^x dW_t, \tag{5.2}$$

which, since $|e^{\eta t}\nabla_x\psi\nabla_x X_t^x|$ is bounded, has a unique solution such that U^x is bounded and V^x belongs to $L^2_{\mathcal{P},\mathrm{loc}}(\Omega;L^2(0,\infty;\Xi^*))$ (see [3] and [21]).

We know that for a suitable sequence $\alpha_n\searrow0$,

$$\bar v^{\alpha_n}(x)=Y_0^{x,\alpha_n}-Y_0^{0,\alpha_n}\to\bar Y_0^x,$$

and we claim now that

$$\nabla\bar v^{\alpha_n}(x)=\nabla Y_0^{x,\alpha_n}=U_0^{x,\alpha_n}\to U_0^x.$$

To prove this we introduce the finite horizon equations: for $t \in [0, N]$,

$$\begin{cases} -dU_t^{x,\alpha,N} = [e^{\eta t}\nabla_x\psi(X_t^x, Z_t^{x,\alpha})\nabla X_t^x - (\alpha+\eta)U_t^{x,\alpha,N} + \nabla_z\psi(X_t^x, Z_t^{x,\alpha})V_t^{x,\alpha,N}]dt \\ \qquad\qquad -V_t^{x,\alpha,N}dW_t, \\ U_N^{x,\alpha,N} = 0. \end{cases}$$

$$\begin{cases} -dU_t^{x,N} = [e^{\eta t}\nabla_x\psi(X_t^x, \bar{Z}_t^x)\nabla X_t^x - \eta U_t^{x,N} + \nabla_z\psi(X_t^x, \bar{Z}_t^x)V_t^{x,N}]dt - V_t^{x,N}dW_t, \\ U_N^{x,N} = 0. \end{cases}$$

Since $\mathbb{E}\int_0^N |Z_s^{x,\alpha_n} - \bar{Z}_s^x|^2 ds \to 0$ it is easy to verify that, for all fixed $N > 0$, $U_0^{x,\alpha_n,N} \to U_0^{x,N}$.

On the other side a standard application of Girsanov's lemma gives (see [16])

$$|U_0^{x,\alpha_n,N} - U_0^{x,\alpha_n}| \le \frac{c}{\alpha_n+\eta}e^{-\eta N}, \qquad |U_0^{x,N} - U_0^x| \le \frac{c}{\eta}e^{-\eta N}$$

for a suitable constant c.

Thus a standard argument implies $U_0^{x,\alpha_n} \to U_0^x$. An identical argument also ensures continuity of U_0^x with respect to x (also taking into account 4.8). The proof is therefore completed. $\square$

As usual in the theory of Markovian BSDEs, the differentiability property allows us to identify the process $\bar{Z}^x$ as a function of the process X^x. To deal with our Banach space setting, we need to make the following extra assumption.

HYPOTHESIS 5.2. *There exists a Banach space Ξ_0, densely and continuously embedded in Ξ, such that $G(\Xi_0) \subset E$ and $G : \Xi_0 \to E$ is continuous.*

We note that this condition is satisfied in most applications. In particular it is trivially true in the special case $E = H$ just by taking $\Xi_0 = \Xi$, since G is assumed to be a linear bounded operator from Ξ to H. The following is proved in [17, Theorem 3.17].

THEOREM 5.3. *Assume that Hypotheses* 3.1, 4.1, *and* 5.2 *hold. Moreover assume that F is of class $\mathcal{G}^1(E, E)$ with ∇F bounded on bounded subsets of E and ψ is of class $\mathcal{G}^1(E \times \Xi^*, E)$. Then $\bar{Z}_t^x = \nabla\bar{v}(X_t^x)G$, $\mathbb{P}$-a.s. for a.e. $t \ge 0$.*

Remark 5.4. We notice that $\nabla\bar{v}(x)G\xi$ is only defined for $\xi \in \Xi_0$ in general, and the conclusion of Theorem 5.3 should be stated more precisely as follows: for $\xi \in \Xi_0$ the equality $Z_t^x\xi = \nabla\bar{v}(X_t^x)G\xi$ holds $\mathbb{P}$-a.s. for a.e. $t \ge 0$. However, since $\bar{Z}^x$ is a process with values in Ξ^*, and more specifically a process in $L^2_{\mathcal{P}}(\Omega, L^2([0,T], \Xi^*))$, it follows that $\mathbb{P}$-a.s. and for a.e. t the operator $\xi \to \nabla\bar{v}(X_t^x)G\xi$ can be extended to a bounded linear operator defined on the whole Ξ. Equivalently, for a.e. t and for almost all $x \in E$ (with respect to the law of X_t) the linear operator $\xi \to \nabla\bar{v}(x)G\xi$ can be extended to a bounded linear operator defined on the whole Ξ (see also Remark 3.18 in [17]).

Remark 5.5. The above representation together with the fact that $\bar{v}$ is Lipschitz with Lipschitz constant $K_x\eta^{-1}$ immediately implies that if F is of class $\mathcal{G}^1(E, E)$ and ψ is of class $\mathcal{G}^1(E \times \Xi^*, E)$, then $|\bar{Z}_t^x|_{\Xi_0^*} \le K_x\eta^{-1}|G|_{L(\Xi_0,E)}$ for all $x \in E$, $\mathbb{P}$-a.s. for a.e. $t \ge 0$. Consequently we can construct $\bar{\zeta}$ in Theorem 4.4 in such a way that it is bounded in the Ξ_0^* norm by $K_x\eta^{-1}|G|_{L(\Xi_0,E)}$.

Once this is proved we can extend the result to the case in which ψ is no longer differentiable but only Lipschitz, namely we can prove that even in this case the process $\bar{Z}^x$ is bounded. Indeed if we consider a sequence $\{\psi_n : n \in \mathbb{N}\}$ of functions of class $\mathcal{G}^1(E \times \Xi^*, E)$ such that for all $x, x' \in H$, $z, z' \in \Xi^*$, $n \in \mathbb{N}$,

$$|\psi_n(x, z) - \psi_n(x', z')| \leq K_x|x - x'| + K_z|z - z'|; \quad \lim_{n\to\infty} |\psi_n(x, z) - \psi(x, z)| = 0.$$

We know that if $(\bar{Y}^{x,n}, \bar{Z}^{x,n}, \lambda_n)$ is the solution of the EBSDE (4.1) with ψ replaced by ψ_n, then $|\bar{Z}_t^{x,n}|_{\Xi_0^*} \leq K_x\eta^{-1}|G|_{L(\Xi_0,E)}$. Then, as we did above, we can show (showing that the corresponding equations with monotonic generator converge uniformly in α) that $\mathbb{E}\int_0^T |\bar{Z}_t^{x,n} - \bar{Z}_t^x|_{\Xi_0^*}^2 dt \to 0$, and the claim follows.

We also notice that by the same argument we also have $|\bar{\zeta}^\alpha(x)|_{\Xi_0^*} \leq K_x\eta^{-1} |G|_{L(\Xi_0,E)}$ for all $\alpha > 0$. Now we introduce the Kolmogorov semigroup corresponding to X: for measurable and bounded $\phi : E \to \mathbb{R}$ we define

$$P_t[\phi](x) = \mathbb{E}\,\phi(X_t^x), \qquad t \geq 0,\, x \in E. \tag{5.3}$$

Definition 5.6. *The semigroup $(P_t)_{t\geq 0}$ is called strongly Feller if for all $t > 0$ there exists k_t such that for all measurable and bounded $\phi : E \to \mathbb{R}$,*

$$|P_t[\phi](x) - P_t[\phi](x')| \leq k_t\|\phi\|_0|x - x'|, \qquad x, x' \in E,$$

where $\|\phi\|_0 = \sup_{x\in E} |\phi(x)|$.

This terminology is somewhat different from the classical one (namely, that P_t maps measurable bounded functions into continuous ones for all $t > 0$), but it will be convenient for us.

Definition 5.7. *We say that F is genuinely dissipative if there exist $\epsilon > 0$ and $c > 0$ such that, for all $x, x' \in E$, there exists $z^* \in \partial|x - x'|$ such that $< z^*, F(x) - F(x') >_{E^*,E} \leq -c|x - x'|^{1+\epsilon}$.*

An example of a genuinely dissipative function is given at the end of section 9.

Lemma 5.8. *Assume that Hypotheses 3.1 and 3.4 hold. If the Kolmogorov semigroup (P_t) is strongly Feller, then for all bounded measurable $\phi : E \to \mathbb{R}$, and all $t \geq 0$,*

$$\left|P_t[\phi](x) - \int_E \phi(x)\mu(dx)\right| \leq ce^{-\eta(t/4)}(1 + |x|)\|\phi\|_0.$$

If in addition F is genuinely dissipative, then for all bounded measurable $\phi : E \to \mathbb{R}$, and all $t \geq 0$,

$$\left|P_t[\phi](x) - \int_E \phi(x)\mu(dx)\right| \leq ce^{-\eta(t/4)}\|\phi\|_0.$$

Proof. If t is small (say $t \leq 2$), there is nothing to prove. For $t > 2$ we have, by Theorem 3.7,

$$\begin{aligned}\left|P_t[\phi](x) - \int_E \phi(x)\mu(dx)\right| &= \left|P_{t-1}[P_1[\phi]](x) - \int_E P_1[\phi](x)\mu(dx)\right| \\ &\leq C(1 + |x|)e^{-\eta t/4}\|P_1[\phi]\|_{\text{lip}} \leq C(1 + |x|)e^{-\eta t/4}k_1\|\phi\|_0,\end{aligned}$$

and the first claim follows for all $t \geq 0$, since $\left|P_t[\phi](x) - \int_E \phi(x)\mu(dx)\right| \leq 2\|\phi\|_0$.

If F is now genuinely dissipative, then in [7, Theorem 6.4.1] it is shown that

$$\left|\mathbb{E}\phi(X_t^x) - \int_E \phi\, d\mu\right| \leq Ce^{-\eta t/2}\|\phi\|_{\text{lip}},$$

and the second claim follows by the same argument. ∎

We are now able to state and prove two corollaries of Theorems 5.1 and 5.3.

COROLLARY 5.9. *Assume that Hypotheses* 3.1, 3.4, 4.1, *and* 5.2 *hold. Moreover assume that F is of class $\mathcal{G}^1$ with ∇F bounded on bounded subsets of E, and that ψ is bounded on each set $E \times B$, where B is any ball of Ξ_0^*. Finally assume that the Kolmogorov semigroup (P_t) is strongly Feller.*

Then the following holds:

$$\lambda = \int_E \psi(x, \bar{\zeta}(x))\mu(dx),$$

where μ is the unique invariant measure of X.

Proof. First notice that $\overline{\psi} := \psi(\,\cdot\,, \bar{\zeta}(\,\cdot\,))$ is bounded by Remark 5.5. Then

$$T^{-1}\mathbb{E}[\bar{Y}_0^x - \bar{Y}_T^x] = T^{-1}\mathbb{E}\int_0^T \left(\psi(X_t^x, \bar{\zeta}(X_t^x)) - \int_E \bar{\psi}\, d\mu\right) dt + \left(\int_E \bar{\psi}\, d\mu - \lambda\right).$$

We know that $T^{-1}\mathbb{E}[\bar{Y}_0^x - \bar{Y}_T^x] \to 0$ by the argument in Theorem 4.6. Moreover by the first conclusion of Lemma 5.8

$$T^{-1}\mathbb{E}\int_0^T \left(\psi(X_t^x, \bar{\zeta}(X_t^x)) - \int_E \bar{\psi}\, d\mu\right) dt \to 0,$$

and the claim follows. ∎

COROLLARY 5.10. *In addition to the assumptions of Corollary* 5.9, *suppose that F is genuinely dissipative. Then $\bar{v}$ is bounded.*

Proof. Let $(Y^{x,\alpha}, Z^{x,\alpha})$ be the solution of (4.2). We know that $Y_t^{x,\alpha} = v^\alpha(X_t^x)$ and $Z_t^{x,\alpha} = \zeta^\alpha(X_t^x)$ with v^α Lipschitz uniformly with respect to α and ζ^α bounded in Ξ^* uniformly with respect to α. Let $\psi^\alpha = \psi(\,\cdot\,, \bar{\zeta}^\alpha(\,\cdot\,))$. Under the present assumptions we also conclude that the maps ψ^α as well are bounded in Ξ^* uniformly with respect to α.

Computing $d(e^{-\alpha t}\bar{Y}_t^{x,\alpha})$, we obtain

$$Y_0^{x,\alpha} = \mathbb{E}e^{-\alpha T}Y_T^{x,\alpha} + \mathbb{E}\int_0^T e^{-\alpha t}\psi^\alpha(X_t^x)dt,$$

and for $T \to \infty$,

$$Y_0^{x,\alpha} = \mathbb{E}\int_0^\infty e^{-\alpha t}\psi^\alpha(X_t^x)dt.$$

Subtracting $\alpha^{-1}\int_E \psi^\alpha(x)\mu(dx)$ from both sides, we obtain

$$\begin{aligned}\left|Y_0^{x,\alpha} - \alpha^{-1}\int_E \psi^\alpha(x)\mu(dx)\right| &= \left|\int_0^\infty e^{-\alpha t}\left[P_t[\psi^\alpha](x) - \int_E \psi^\alpha(x)\mu(dx)\right] dt\right| \\ &\le 4c\eta^{-1}\|\psi^\alpha\|_0,\end{aligned}$$

where the last inequality comes from the second conclusion of Lemma 5.8.

Thus $|Y_0^{x,\alpha} - Y_0^{0,\alpha}| \le 8c\eta^{-1}\|\psi^\alpha\|_0$ and the claim follows since by construction $Y_0^{x,\alpha} - Y_0^{0,\alpha} \to \bar{v}(x)$. ∎

6. Ergodic Hamilton–Jacobi–Bellman equations. We briefly show here that if $\bar{Y}_0^x = \bar{v}(x)$ is of class $\mathcal{G}^1$, then the couple (v, λ) is a mild solution of the following "ergodic" HJB equation:

$$\mathcal{L}v(x) + \psi(x, \nabla v(x)G) = \lambda, \quad x \in E, \tag{6.1}$$

where linear operator $\mathcal{L}$ is formally defined by

$$\mathcal{L}f(x) = \frac{1}{2}Trace\left(GG^*\nabla^2 f(x)\right) + \langle Ax, \nabla f(x)\rangle_{E,E^*} + \langle F(x), \nabla f(x)\rangle_{E,E^*}.$$

We notice that we can define the transition semigroup $(P_t)_{t\geq 0}$ corresponding to X by the formula (5.3) for all measurable functions $\phi : E \to \mathbb{R}$ having polynomial growth, and we notice that $\mathcal{L}$ is the formal generator of $(P_t)_{t\geq 0}$.

Since we are dealing with an elliptic equation, it is natural to consider (v, λ) as a mild solution of (6.1) if and only if for arbitrary $T > 0$, $v(x)$ coincides with the mild solution $u(t, x)$ of the corresponding parabolic equation having v as a terminal condition:

$$\begin{cases} \frac{\partial u(t,x)}{\partial t} + \mathcal{L}u(t,x) + \psi(x, \nabla u(t,x)G) - \lambda = 0, \quad t \in [0,T],\ x \in E, \\ u(T,x) = v(x), \quad x \in E. \end{cases} \tag{6.2}$$

Thus we are led to the following definition (see also [14]).

DEFINITION 6.1. *A pair* (v, λ) *(*$v : E \to \mathbb{R}$ *and* $\lambda \in \mathbb{R}$*) is a mild solution of the HJB equation* (6.1) *if the following are satisfied:*

1. $v \in \mathcal{G}^1(E, \mathbb{R})$,
2. *there exists* $C > 0$ *such that* $|\nabla v(x)h| \leq C|h|_E\left(1 + |x|_E^k\right)$ *for every* $x, h \in E$ *and some positive integer* k,
3. *for* $0 \leq t \leq T$ *and* $x \in E$,

$$v(x) = P_{T-t}[v](x) + \int_t^T \left(P_{s-t}[\psi(\cdot, \nabla v(\cdot)G)](x) - \lambda\right) ds. \tag{6.3}$$

In the right-hand side of (6.3) we notice the occurrence of the term $\nabla v(\cdot)G$, which is not well defined as a function $E \to \Xi^*$, since G is not required to map Ξ into E. The situation is similar to Remark 5.4. In general, for $x \in E$, $\nabla v(x)G\xi$ is only defined for $\xi \in \Xi_0$. In (6.3) it is implicitly required that $\mathbb{P}$-a.s., and for a.e. t, the operator $\xi \to \nabla v(X_t^x)G\xi$ can be extended to a bounded linear operator defined on the whole Ξ. Noting that

$$P_t[\psi(\cdot, \nabla v(\cdot)G)](x) = \mathbb{E}\,\psi(X_t^x, \nabla v(X_t^x)G),$$

(6.3) is now meaningful.

Using the results for the parabolic case (see [17]) we get existence of the mild solution of (6.1) whenever we have proved that the function $\bar{v}$ in Theorem 4.4 is differentiable.

THEOREM 6.2. *Assume that Hypotheses* 3.1, 4.1*, and* 5.2 *hold. Moreover assume that* F *is of class* $\mathcal{G}^1(E, E)$ *with* ∇F *bounded on bounded subsets of* E *and* ψ *is of class* $\mathcal{G}^1(E \times \Xi^*, E)$*.*

Then $(\bar{v}, \bar{\lambda})$ *is a mild solution of the HJB equation* (6.1)*.*

Conversely, if (v, λ) *is a mild solution of* (6.1)*, then, setting* $Y_t^x = v(X_t^x)$ *and* $Z_t^x = \nabla v(X_t^x)G$*, the triple* (Y^x, Z^x, λ) *is a solution of the EBSDE* (4.1)*.*

7. Optimal ergodic control. Assume that Hypothesis 3.1 holds and let X^x denote the solution to (3.1). Let U be a separable metric space. We define a control u as an $(\mathcal{F}_t)$-progressively measurable U-valued process. The cost corresponding to a given control is defined in the following way. We assume that the functions $R : U \to \Xi^*$ and $L : E \times U \to \mathbb{R}$ are measurable and satisfy, for some constant $c > 0$,

(7.1)
$$|R(u)| \leq c, \quad |L(x,u)| \leq c, \quad |L(x,u) - L(x',u)| \leq c\,|x - x'|, \qquad u \in U,\, x, x' \in E.$$

Given an arbitrary control u and $T > 0$, we introduce the Girsanov density

$$\rho_T^u = \exp\left(\int_0^T R(u_s)dW_s - \frac{1}{2}\int_0^T |R(u_s)|^2_{\Xi^*}ds\right)$$

and the probability $\mathbb{P}_T^u = \rho_T^u\mathbb{P}$ on $\mathcal{F}_T$. The ergodic cost corresponding to u and the starting point $x \in E$ is

$$J(x,u) = \limsup_{T\to\infty} \frac{1}{T}\mathbb{E}^{u,T}\int_0^T L(X_s^x, u_s)ds, \tag{7.2}$$

where $\mathbb{E}^{u,T}$ denotes expectation with respect to $\mathbb{P}_T^u$. We notice that $W_t^u = W_t - \int_0^t R(u_s)ds$ is a Wiener process on $[0,T]$ under $\mathbb{P}_T^u$ and that

$$dX_t^x = (AX_t^x + F(X_t^x))dt + G(dW_t^u + R(u_t)dt), \quad t \in [0,T],$$

which justifies our formulation of the control problem. Our purpose is to minimize the cost over all controls.

To this purpose we first define the Hamiltonian in the usual way

$$\psi(x,z) = \inf_{u\in U}\{L(x,u) + zR(u)\}, \qquad x \in E,\, z \in \Xi^*, \tag{7.3}$$

and we remark that if, for all x, z, the infimum is attained in (7.3), then there exists a measurable function $\gamma : E \times \Xi^* \to U$ such that

$$\psi(x,z) = l(x,\gamma(x,z)) + zR(\gamma(x,z)).$$

This follows from an application of Theorem 4 of [19].

We notice that under the present assumptions ψ is a Lipschitz function and $\psi(\cdot,0)$ is bounded (here the fact that R depends only on u is used). So if we assume Hypotheses 3.1 and 3.4, then in Theorem 4.4 we have constructed, for every $x \in E$, a triple

$$(\bar{Y}^x, \bar{Z}^x, \bar{\lambda}) = (\bar{v}(X^x), \bar{\zeta}(X^x), \bar{\lambda}) \tag{7.4}$$

solution to the EBSDE (4.1).

THEOREM 7.1. *Assume that Hypotheses* 3.1, 3.4, *and* 3.5 *hold and that* (7.1) *holds as well.*

Moreover suppose that, for some $x \in E$, a triple (Y, Z, λ) verifies $\mathbb{P}$-a.s. (4.1) *for all $0 \leq t \leq T$, where Y is a progressively measurable continuous process, Z is a process in $L^2_{\mathcal{P},\mathrm{loc}}(\Omega; L^2(0,\infty;\Xi^*))$, and $\lambda \in \mathbb{R}$. Finally assume that there exists $c_x > 0$ (that may depend on x) such that $\mathbb{P}$-a.s.*

$$|Y_t| \leq c_x(|X_t^x| + 1) \text{ for all } t \geq 0.$$

Then the following holds:

(i) *For arbitrary control* u *we have* $J(x,u) \geq \lambda = \bar{\lambda}$, *and the equality holds if and only if* $L(X_t^x, u_t) + Z_t R(u_t) = \psi(X_t^x, Z_t)$, $\mathbb{P}$*-a.s. for a.e.* t.

(ii) *If the infimum is attained in* (7.3), *then the control* $\bar{u}_t = \gamma(X_t^x, Z_t)$ *verifies* $J(x, \bar{u}) = \bar{\lambda}$.

In particular, for the solution (7.4) *mentioned above, we have the following:*

(iii) *For arbitrary control* u *we have* $J(x,u) = \bar{\lambda}$ *if and only if* $L(X_t^x, u_t) + \bar{\zeta}(X_t^x) R(u_t) = \psi(X_t^x, \bar{\zeta}(X_t^x))$, $\mathbb{P}$*-a.s. for a.e.* t.

(iv) *If the infimum is attained in* (7.3), *then the control* $\bar{u}_t = \gamma(X_t^x, \bar{\zeta}(X_t^x))$ *verifies* $J(x, \bar{u}) = \bar{\lambda}$.

Remark 7.2.

1. The equality $\lambda = \bar{\lambda}$ clearly follows from Theorem 4.6.
2. Points (iii) and (iv) are immediate consequences of (i) and (ii).
3. The conclusion of point (iv) is that there exists an optimal control in feedback form, with the optimal feedback given by the function $x \mapsto \gamma(x, \bar{\zeta}(x))$.
4. Under the conditions of Theorem 6.2, the pair $(\bar{v}, \bar{\lambda})$ occurring in (7.4) is a mild solution of the HJB equation (6.1).
5. It follows from the proof below that if lim sup is changed into lim inf in the definition (7.2) of the cost, then the same conclusions hold, with the obvious modifications, and the optimal value is given by $\bar{\lambda}$ in both cases.

Proof. As $(Y, Z, \bar{\lambda})$ is a solution of the ergodic BSDE, we have

$$\begin{aligned} -dY_t &= [\psi(X_t^x, Z_t) - \bar{\lambda}]dt - Z_t dW_t \\ &= [\psi(X_t^x, Z_t) - \bar{\lambda}]dt - Z_t dW_t^u - Z_t R(u_t)dt, \end{aligned}$$

from which we deduce that

$$\begin{aligned} \bar{\lambda} = \frac{1}{T}\mathbb{E}^{u,T}[Y_T - Y_0] &+ \mathbb{E}^{u,T}\frac{1}{T}\int_0^T [\psi(X_t^x, Z_t) - Z_t R(u_t) - L(X_t^x, u_t)]dt \\ &+ \frac{1}{T}\mathbb{E}^{u,T}\int_0^T L(X_t^x, u_t)dt. \end{aligned}$$

Thus

$$\frac{1}{T}\mathbb{E}^{u,T}\int_0^T L(X_t^x, u_t)dt \geq \frac{1}{T}\mathbb{E}^{u,T}[Y_0 - Y_T] + \bar{\lambda}.$$

But by (3.6) we have

$$|\mathbb{E}^{u,T} Y_T| \leq c\mathbb{E}^{u,T}(|X_T^x| + 1) \leq c(1 + |x|).$$

Consequently $T^{-1}\mathbb{E}^{u,T}[Y_0 - Y_T] \to 0$, and

$$\limsup_{T\to\infty} \frac{1}{T}\mathbb{E}^{u,T}\int_0^T L(X_t^x, u_t)dt \geq \bar{\lambda}.$$

Similarly, if $L(X_t^x, u_t) + Z_t R(u_t) = \psi(X_t^x, Z_t)$,

$$\frac{1}{T}\mathbb{E}^{u,T}\int_0^T L(X_t^x, u_t)dt = \frac{1}{T}\mathbb{E}^{u,T}[Y_0 - Y_T] + \bar{\lambda},$$

and the claim holds. $\square$

8. Uniqueness. We wish now to adapt the argument in [15] in order to obtain uniqueness of solutions to the EBSDE when we impose that $Y_t = v(X_t^x)$, $Z_t = \zeta(X_t^x)$. The argument requires that the Markov process related to the state equation with continuous feedback enjoys recurrence properties.

In this section, beside the general assumptions in Hypotheses 3.1 and 4.1, we suppose that

$$E = H \qquad \text{and} \qquad F \text{ is bounded.} \tag{8.1}$$

We recall a result due to [22] on recurrence of solution to SDEs.

THEOREM 8.1. *Consider*

$$dX_t = (AX_t + g(X_t))dt + GdW_t, \tag{8.2}$$

where $g : H \to H$ *is bounded and weakly continuous (that is* $x \to \langle \xi, g(x)\rangle$ *is continuous for all* $\xi \in H$*). Let*

$$Q_t = \int_0^t e^{sA} GG^* e^{sA^*} ds,$$

and assume the following:

1. $\sup_{t\geq 0}$ *Trace* $(Q_t) < \infty$*,*
2. Q_t *is injective for* $t > 0$*,*
3. $e^{tA}(H) \subset (Q_t)^{1/2}(H)$ *for* $t > 0$*,*
4. $\int_0^t |Q_s^{-1/2} e^{sA}| ds < \infty$ *for* $t > 0$*,*
5. *there exists* $\beta > 0$ *such that* $\int_0^t s^{-\beta}$ *Trace* $(S(s)S(s)^*)\, ds < \infty$ *for* $t > 0$*.*

Then, for all $T > 0$*,* (8.2) *admits a martingale solution on* $[0, T]$*, unique in law. The associated transition probabilities* $P(t, x, T, \cdot)$ *on* H *(*$0 \leq t \leq T, x \in H$*) identify a recurrent Markov process on* $[0, \infty)$*.*

Consider now the ergodic control problem with state equation

$$dX_t^{x,u} = (AX_t^{x,u} + F(X_t^{x,u}) + GR(u_t))dt + GdW_t, \qquad X_0^{x,u} = x,$$

and cost

$$\limsup_{T\to\infty} \frac{1}{T}\, \mathbb{E} \int_0^T L(X_s, u_s) ds,$$

where $R : U \to \Xi$ is continuous and bounded.

We restrict ourselves to the class of controls given by continuous feedbacks, i.e., given an arbitrary continuous $u : H \to U$ (called feedback) we define the corresponding trajectory as the solution of

$$dX_t^{x,u} = (AX_t^{x,u} + F(X_t^{x,u}))dt + G(R(u(X_t^{x,u}))dt + dW_t), \qquad X_0^{u,x} = x.$$

We notice that for all $T > 0$ there exists a weak solution $X^{x,u}$ of this equation, and it is unique in law.

We set as usual

$$\psi(x, z) = \inf_{u\in U} \{L(x, u) + zR(u)\},$$

and assume that ψ is continuous and there exists a continuous $\gamma : H \times \Xi \to U$ such that

$$\psi(x, z) = L(x, \gamma(x, z)) + zR(\gamma(x, z)).$$

THEOREM 8.2. *Suppose* (8.1) *and suppose that the assumptions of Theorem* 8.1 *hold. Let* (v, ζ, λ) *with* $v : H \to \mathbb{R}$ *continuous,* $\zeta : H \to \mathbb{R}$ *continuous, and* λ *a real number satisfy the following conditions:*

1. $|v(x)| \leq c|x|$*;*
2. *for an arbitrary filtered probability space with a Wiener process* $(\hat{\Omega}, \hat{\mathcal{F}}, \{\hat{\mathcal{F}}_t\}_{t\geq 0}, \hat{\mathbb{P}}, \{\hat{W}_t\}_{t\geq 0})$ *and for any solution of*

$$d\hat{X}_t = (A\hat{X}_t + F(\hat{X}_t))dt + Gd\hat{W}_t, \qquad t \in [0,T],$$

setting $Y_t = v(\hat{X}_t),\ Z_t = \zeta(\hat{X}_t)$*, we have*

$$-dY_t = [\psi(\hat{X}_t, Z_t) - \lambda]dt - Z_t dW_t, \quad t \in [0,T].$$

Let

$$\tau_r^T = \inf\{s \in [0,T] : |X_s^{u,x}| < r\},$$

with the convention $\tau_r^T = T$ *if the indicated set is empty, and*

$$J(x,u) = \limsup_{r\to 0} \limsup_{T\to\infty} \mathbb{E}\int_0^{\tau_r^T} [\psi(X_s^{x,u}, \zeta(X_s^{x,u})) - \lambda]ds.$$

Then

$$v(x) = \inf_u J(x,u),$$

where the infimum (that is a minimum) is taken over all continuous feedbacks u*.*

Proof. Let $u : H \to U$ be continuous. We notice that $X^{x,u}$ solves on $[0,T]$:

$$dX_t^{x,u} = (AX_t^{x,u} + F(X_t^{x,u}))dt + Gd\tilde{W}_t^u,\ t \in [0,T],$$

where $\tilde{W}_t = \int_0^t R(u(X_r^{x,u}))dr + W_t$ is a Wiener process on $[0,T]$ under a suitable probability $\hat{\mathbb{P}}^{u,T}$.

Therefore $Y_t = v(X_t^{x,u}),\ Z_t = \zeta(X_t^{x,u})$ satisfy

$$-dY_t = [\psi(X_t^{x,u}, \zeta(X_t^{x,u})) - \lambda]dt - Z_t R(u(X_t^{x,u}))dt - Z_t dW_t.$$

Integrating in $[0, \tau_r^T]$ we get

$$v(x) = \mathbb{E}(v(X_{\tau_r^T}^{x,u})) + \mathbb{E}\int_0^{\tau_r^T} [\psi(X_s^{u,x}, \zeta(X_s^{x,u})) - \lambda - Z_s R(u(X_s^{x,u}))]ds.$$

Thus,

$$v(x) \leq \mathbb{E}(v(X_{\tau_r^T}^{x,u})) + \mathbb{E}\int_0^{\tau_r^T} [L(X_s^{u,x}, u(X_s^{x,u})) - \lambda]ds. \tag{8.3}$$

Now, since $\sup_{T\geq 0} \mathbb{E}|X_T^{x,u}|^2 < \infty$ (recall that for $\eta > 0$, $A + F + \eta I$ is dissipative and R and G are bounded), we have

$$\begin{aligned} |\mathbb{E}(v(X_{\tau_r^T}^{x,u}))| \leq c\mathbb{E}|X_{\tau_r^T}^{x,u}| &\leq cr + (\mathbb{E}(|X_T^{x,u}|^2))^{1/2}(\mathbb{P}(\tau_r^T = T))^{1/2} \\ &\leq cr + c(\mathbb{P}(\tau_r^T = T))^{1/2}. \end{aligned}$$

Notice that $\mathbb{P}(\tau_r^T = T) = \tilde{\mathbb{P}}(\inf_{t\in[0,T]} |\tilde{X}_t| \geq r)$, where $\tilde{X}$ is the Markov process on the whole $[0, +\infty)$ corresponding to (8.2) with $g = F(\cdot) + GR(u(\cdot))$.

Since $\tilde{X}$ is recurrent, for all $r > 0$, it holds that $\tilde{\mathbb{P}}(\inf_{t\in[0,T]} |\tilde{X}_t| > r) \to 0$ as $T \to \infty$. Thus

$$\limsup_{r\to 0} \limsup_{T\to\infty} |\mathbb{E}(v(X^{x,u}_{\tau_r^T}))| \to 0.$$

Hence,

$$v(x) \leq \limsup_{r\to 0} \limsup_{T\to\infty} \mathbb{E} \int_0^{\tau_r^T} [L(X_s^{x,u}, u(X_s^{x,u})) - \lambda] ds.$$

The proof is completed noticing that if u is chosen as $u(x) = \gamma(x, \zeta(x))$, then the above inequality becomes an equality. $\square$

This result, combined with Theorems 4.6 and 6.2, gives the following corollary.

Corollary 8.3. *Suppose that all the assumptions of Theorems* 4.6, 6.2, *and* 8.2 *hold. Then* $(\bar{v}, \bar{\lambda})$ *is the unique mild solution of the HJB equation* (6.1) *satisfying* $|\bar{v}(x)| \leq c|x|$.

9. Application to ergodic control of a semilinear heat equation. In this section we show how our results can be applied to perform the synthesis of the ergodic optimal control when the state equation is a semilinear heat equation with additive noise. More precisely, we treat a stochastic heat equation in space dimension one with a dissipative nonlinear term and with control and noise acting on a subinterval. We consider homogeneous Dirichlet boundary conditions.

In $(\Omega, \mathcal{F}, \mathbb{P})$ with a filtration $(\mathcal{F}_t)_{t\geq 0}$ satisfying the usual conditions, we consider, for $t \in [0, T]$ and $\xi \in [0, 1]$, the following equation:

$$(9.1) \qquad \begin{cases} d_t X^u(t,\xi) = \left[\frac{\partial^2}{\partial \xi^2} X^u(t,\xi) + f(\xi, X^u(t,\xi)) + \chi_{[a,b]}(\xi) u(t,\xi)\right] dt \\ \qquad\qquad\qquad + \chi_{[a,b]}(\xi) \dot{W}(t,\xi)\, dt, \\ X^u(t,0) = X^u(t,1) = 0, \\ X^u(0,\xi) = x_0(\xi), \end{cases}$$

where $\chi_{[a,b]}$ is the indicator function of $[a, b]$ with $0 \leq a \leq b \leq 1$; $\dot{W}(t, \xi)$ is a space-time white noise on $[0, T] \times [0, 1]$.

We introduce the cost functional

$$(9.2) \qquad J(x,u) = \limsup_{T\to\infty} \frac{1}{T} \mathbb{E} \int_0^T \int_0^1 l(\xi, X_s^u(\xi), u_s(\xi))\, \mu(d\xi)\, ds,$$

where μ is a finite Borel measure on $[0, 1]$. An admissible control $u(t, \xi)$ is a predictable process such that, for all $t \geq 0$, $\mathbb{P}$-a.s. $u(t, \cdot) \in U := \{v \in C([0, 1]) : |v(\xi)| \leq \delta\}$. We denote by $\mathcal{U}$ the set of such admissible controls. We wish to minimize the cost over $\mathcal{U}$, adopting the formulation of section 7, i.e., by a change of probability in the form of (7.2). The cost introduced in (9.2) is well defined on the space of continuous functions on the interval $[0, 1]$, but for an arbitrary μ it is not well defined on the Hilbert space of square integrable functions.

We suppose the following hypothesis.

Hypothesis 9.1.

1. $f : [0, 1] \times \mathbb{R} \to \mathbb{R}$ *is continuous, and for every* $\xi \in [0, 1]$, $f(\xi, \cdot)$ *is decreasing. Moreover there exist* $C > 0$ *and* $m > 0$ *such that for every* $\xi \in [0, 1], x \in \mathbb{R}$,

$$|f(\xi, x)| \leq C(1 + |x|)^m, \qquad f(0, x) = f(1, x) = 0.$$

2. $l : [0,1]\times\mathbb{R}\times[-\delta,\delta] \to \mathbb{R}$ *is continuous and bounded, and* $l(\xi,\cdot,u)$ *is Lipschitz continuous uniformly with respect to* $\xi \in [0,1]$*,* $u \in [-\delta,\delta]$*.*
3. $x_0 \in C([0,1])$*,* $x_0(0) = x_0(1) = 0$*.*

To rewrite the problem in an abstract way, we set $H = \Xi = L^2(0,1)$ and $E = C_0([0,1]) = \{y \in C([0,1]) : y(0) = y(1) = 0\}$. We define an operator A in E by

$$D(A) = \{y \in C^2([0,1]) : y, y'' \in C_0([0,1])\}, \qquad (Ay)(\xi) = \frac{\partial^2}{\partial \xi^2} y(\xi) \text{ for } y \in D(A).$$

We notice that A is the generator of a C_0 semigroup in E, admitting an extension to H, and $\left|e^{tA}\right|_{L(E,E)} \le e^{-t}$; see, for instance, Theorem 11.3.1 in [7]. As a consequence, $A+F+I$ is dissipative in E.

We set, for $x \in E$, $\xi \in [0,1]$, $z \in \Xi$, $u \in U$,

(9.3)
$$F(x)(\xi) = f(\xi, x(\xi)), \quad (Gz)(\xi) = \chi_{[a,b]}(\xi) z(\xi), \quad L(x,u) = \int_0^1 l(\xi, x(\xi), u(\xi))\, \mu(d\xi),$$

and let R denote the canonical embedding of $C([0,1])$ in $L^2(0,1)$.

Finally $\{W_t, t \ge 0\}$ is a cylindrical Wiener process in H with respect to the filtration $(\mathcal{F}_t)_{t\ge 0}$.

It is easy to verify that Hypotheses 3.1 and 3.4 are satisfied (for the proof of point 4 in Hypothesis 3.1 and of Hypothesis 3.4; see again [7, Theorem 11.3.1]).

We notice that the assumption $f(0,x) = f(1,x) = 0$ ensures that F maps E into E as we assumed throughout the paper. In the literature sometimes this assumption is replaced by the weaker one requiring only that F maps E into H with suitable regularity (see [7]). We have chosen to work under the stronger assumption because it avoids several technical difficulties and allows us to exploit the results proved in [17].

Moreover (see, for instance, [5]), for some $C > 0$,

$$\left|e^{tA}\right|_{L(H,E)} \le C t^{-1/4}, \qquad t \in (0,1];$$

thus Hypothesis 3.5 holds.

Also Hypothesis 5.2 is satisfied by taking $\Xi_0 = \{f \in C_0([0,1]) : f(a) = f(b) = 0\}$.

Clearly the controlled heat equation (9.1) can now be written in an abstract way in the Banach space E as

$$\text{(9.4)} \qquad \begin{cases} dX_t^{x_0,u} = [AX_t^{x_0,u} + F(X_t^{x_0,u})]\, dt + GRu_t dt + GdW_t, \quad t \in [0,T], \\ X_0^{x_0,u} = x_0, \end{cases}$$

and the results of the previous sections can be applied to the ergodic cost (9.2) (reformulated by a change of probability in the form of (7.2)).

In particular if we define, for all $x \in C_0([0,1])$, $z \in L^2(0,1)$, $u \in U$ (identifying $L^2(0,1)$ with its dual)

$$\psi(x,z) = \inf_{u\in U} \left\{ \int_0^1 l(\xi, x(\xi), u(\xi))\mu(d\xi) + \int_a^b z(\xi) u(\xi) d\xi \right\},$$

then there exist $\overline{v} : E \to \mathbb{R}$ Lipschitz continuous and with $\overline{v}(0) = 0$, $\overline{\zeta} : E \to \Xi^*$ measurable, and $\overline{\lambda} \in \mathbb{R}$ such that if $X^{x_0} = X^{x_0,0}$ is the solution of (9.4), then $(\overline{v}(X^{x_0}), \overline{\zeta}(X^{x_0}), \overline{\lambda})$ is a solution of the EBSDE (4.1), and the characterization of the optimal ergodic control stated in Theorem 7.1 holds (and $\overline{\lambda}$ is unique in the sense of Theorem 4.6).

Moreover if f is of class $C^1(\mathbb{R})$ (consequently F will be of class $\mathcal{G}^1(E,E)$) and ψ is of class $\mathcal{G}^1(E\times\Xi^*,E)$, then by Theorem 5.1, $\overline{v}$ is of class $\mathcal{G}^1(E,E)$ and, by Theorem 6.2, it is a mild solution of the ergodic HJB equation (6.1) and it holds that $\overline{\zeta}=\nabla\overline{v}G$.

Let us then consider the particular case in which $[a,b]=[0,1]$, $f(\xi,x)=f(x)$ is of class C^1 with derivative having polynomial growth, and satisfies $f(0)=0$, $[f(x+h)-f(x)]h\leq -c|h|^{2+\epsilon}$ for suitable $c,\epsilon>0$ and all $x,h\in\mathbb{R}$ (for instance, $f(x)=-x^3$). In that case the Kolmogorov semigroup corresponding to the process X^{x_0} is strongly Feller (see [5] and [18]) and it is easy to verify that F is genuinely dissipative (see Definition 5.7). Moreover we can choose $\Xi_0=C_0([0,1])$, and it turns out that ψ is bounded on each set $E\times B$, where B is any ball of Ξ_0^*. Thus the claims of Corollaries 5.9 and 5.10 hold true, and in particular, $\overline{v}$ is bounded.

Finally if we assume that μ is Lebesgue measure and f is bounded and Lipschitz, we can choose $E=\Xi=\Xi_0=H=L^2(0,1)$. Then the assumptions of Theorem 8.1 are satisfied and we can apply Theorem 8.2 to characterize the function $\overline{v}$. In particular, if f is of class $C^1(\mathbb{R})$ and ψ is of class $\mathcal{G}^1(H\times\Xi^*,H)$, then $\overline{v}$ is the unique mild solution of the ergodic HJB equation (6.1).

Acknowledgments. The authors wish to thank the two anonymous referees for their useful comments.

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